math 251 - week 11 - ch 7

Diagonalization of a matrix

Orthogonal matrix:

If its transpose is same as its inverse.

$$A\,A^T = A^T . A = I \rightarrow \text{Identity matrix.}$$ مصفوفة الوحدة

EX Check whether the matrix is Orthogonal or not.

$$A = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$$

Sol.

$$A^T = \begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix}$$

$$A . A^T = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix} \begin{bmatrix} 3/7 & -6/7 & 2/7 \\ 2/7 & 3/7 & 6/7 \\ 6/7 & 2/7 & -3/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I_3$$

Quadratic forms: two theorems:

$\boxed{1}$ $a_1 x_1^2 + a_2 x_2^2 + 2 a_3 x_1 x_2$

$$\Rightarrow [x_1 \quad x_2] \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x^T A x$$

And $\boxed{2}$

$$a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + 2a_4 x_1 x_2 + 2a_5 x_1 x_3 + 2a_6 x_2 x_3$$

$$\Rightarrow [x_1 \ x_2 \ x_3] \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T A x$$

Ex] Express the Quadratic form in matrix ~~notation~~ notation

ⓐ $2x^2 + \underline{6xy} - 5y^2$ $\longrightarrow$ $2 \times \underline{3}\, xy$

ⓑ $x_1^2 + 7x_2^2 - 3x_3^2 + \underline{4x_1 x_2} - \underline{2x_1 x_3} + \underline{8x_1 x_2}$

$2 \times \underline{2}\, x_1 x_2$

$2 \times \underline{-1}\, x_1 x_3$

$2 \times \underline{4}\, x_1 x_3$

Sol. ⓐ $2x^2 + \underline{6xy} - 5y^2$

$2 \times \underline{3}\, xy$

$$[x \ y] \begin{bmatrix} 2 & 3 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

ⓑ

$$[x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 4 \\ -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

ملاحظة

Quadratic forms $\Rightarrow$ <u>Symmetric matrix</u>

<u>Theorm</u>: If A is a Symmetrix matrix, then

ⓐ $x^T A x$ is a +ve definit iff Eigenvalue of $A > 0$.

ⓑ $x^T A x$ is a −ve definit iff Eigenvalue of $A < 0$.

ⓒ $x^T A x$ is an Indefinit iff at least one +ve and one −ve.
E.value. E.value

<u>Ex</u> Find the <u>nature</u> of the Quadratic forms

ⓐ $x^2+5y^2+z^2+2xy+6yz+2zx$.

ⓑ $3x^2+5y^2+3z^2-2yz+2zx-2xy$.

<u>Sol.</u>

ⓐ $[x\ y\ z]\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

that is $A=\begin{bmatrix}1 & 1 & 3\\ 1 & 5 & 1\\ 3 & 1 & 1\end{bmatrix}$

the characteristic equation:-

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & -1 & -3 \\ -1 & \lambda-5 & -1 \\ -3 & -1 & \lambda-1 \end{vmatrix} = 0$$

$\lambda = \underbrace{-2}_{-ve}, \underbrace{3, 6}_{+ve}$ are the Eigenvalues.

So the given Quadratic form is Indefinit

because Some Eigenvalues are +ve and

Some Eigenvalues are -ve.

* The nature of Quadratic ~~form~~ is Indefinit.

(b) $$[x \;\; y \;\; z] \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

the characteristic equation is

$$|\lambda I - A| = \begin{vmatrix} \lambda-3 & +1 & -1 \\ 1 & \lambda-5 & 1 \\ -1 & 1 & \lambda-3 \end{vmatrix} = 0$$

$\lambda = \underbrace{2, 3, 6}_{+ve}$ are the eigenvalues

the Quadratic form is +ve definit.

* the nature of Quadratic form is +ve (positive) definit.

Conjugate Transpose: (complex numbers)

<u>def.</u> If A is a complex matrix, then $A^* = \bar{A}^T$

$1+i \xrightarrow{\text{conj.}} 1-i$

$-i \xrightarrow{\text{conj.}} i$

$8 \xrightarrow{\text{conj.}} 8 \in \mathbb{R}$

not complex number.

Complex number $\rightarrow a+ib \xrightarrow{\text{conj.}} a-ib$

$6 \xrightarrow{\text{conj.}} 6$

$6 \in \mathbb{R}$ real number.

$\boxed{A^* \Rightarrow \text{Conj. transpose}}$

Ex) find the Conjugate Transpose of A.

$$A = \begin{bmatrix} 1+i & -i & 0 \\ 2 & 3-2i & i \end{bmatrix}$$

Sol.

$$\bar{A} = \begin{bmatrix} 1-i & i & 0 \\ 2 & 3+2i & -i \end{bmatrix}$$

$$\Rightarrow \bar{A}^T = \begin{bmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{bmatrix} = A^*$$

Conjugat transpose.

def. Hermitian matrix $A^{-1} = A^{*}$

The Eigenvalue of hermitian matrix is real.

Ex/ If the matrix $A = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix}$

Is the matrix Orthogonal.

Sol- If $A \cdot A^T = I$, then A is Orthogonal.

$$A^T = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

$$A \cdot A^T = \begin{pmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}$$

$$A \cdot A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Identity matrix of order 3X3

Ex) Express the Quadratic form in the matrix notation $x^T A x$ where A is a symmetric matrix.

(A) $3x_1^2 + 7x_2^2$ (B) $9x_1^2 - x_2^2 + 4x_3^2 + 6x_1x_2 - 8x_1x_3 + \underline{x_2x_3}$

$x_2x_3 \rightarrow 2 \times \frac{1}{2} x_2 x_3$

Sol.

(A) $$[x_1 \quad x_2] \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(B) $$[x_1 \quad x_2 \quad x_3] \begin{bmatrix} 9 & 3 & -4 \\ 3 & -1 & 1/2 \\ -4 & 1/2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$